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THE PHYSICS OF VIBRATING STRINGS

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Vibrating strings are key components of many musical instruments, such as guitars, violins, and pianos. Hence, if we want to understand the physics of these instruments, it is reasonable to begin with the physics of a vibrating string. To a first approximation, all strings are created equal, as all are described by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{1}$$

where y, the displacement of the string from its equilibrium position, is a function of the position x along the string and the time t. Because guitars sound different from violins, and both sound different from pianos, we know that there must be more to the story than (1). This column will explore some of the physics that is important in musical instruments. Broadly speaking, the additional physics falls into two categories: the nature of how the string is excited, and small corrections or additions to the wave equation.

The wave equation (1) can be readily derived from Newton's second law. The parameter c is equal to $\sqrt{T/\mu}$, where T is the tension and μ is the mass per unit length of the string. It is straightforward to show that the solutions to (1) have the form $y = f(x \pm ct)$, so that c is the wave speed. To construct a numerical scheme for solving (1), we discretize x and t in units of Δx and Δt, so that $y(x,t) \to y(i\Delta x, n\Delta t) \to y(i,n)$, and write the derivatives in finite difference form:[1]

$$\frac{y(i,n+1)+y(i,n-1)-2y(i,n)}{(\Delta t)^2}$$

$$\approx c^2 \left[\frac{y(i+1,n)+y(i-1,n)-2y(i,n)}{(\Delta x)^2} \right]. \tag{2}$$

The $\approx$ symbol is used to emphasize that (2) is only approximate; as we shall see shortly, there are correction terms that might be important. We can rearrange (2) as

$$y(i,n+1) = 2(1-r^2)y(i,n) - y(i,n-1) + r^2[(y(i+1,n)+y(i-1,\, n)], \tag{3}$$

where $r \equiv c\Delta t/\Delta x$. From the form of (3) we see that, if we know the string configuration at time steps n and $n-1$, we can calculate the new configuration at time step $n+1$. The initial conditions depend on how the string is excited.

Some results obtained using algorithm (3) are shown in Fig. 1. The initial string profile (shown at the top of Fig. 1) is triangular, with the string at rest, as would be appropriate for a plucked guitar string. That is, we take the string profile at time steps $n=0$ and $n=1$ to be the triangular shape shown at the top in Fig. 1. The simulation begins with the calculation of the string position at the next time step, $n=2$. The kink associated with this plucked profile is seen to split into two separate kinks, one propagating to the left and one to the right, which reflect from the ends of the string. In this simulation the ends of the string are kept fixed, and so the reflections are inverted.

An astute reader will recognize that the form of (3) bears a strong resemblance to a no-frills Euler algorithm. Although the Euler method is a simple and useful approach for many problems, it is known to fail miserably for some situations that involve oscillatory motion, such as a simple harmonic oscillator or planetary motion.[1] Wave motion is also a type of oscillation, and so we might expect that an Euler approach would fail here too. However, the results in Fig. 1 suggest that it works quite well. Understanding why it works so well provides a useful lesson in numerical methods.

Many algorithms for treating differential equations involve a discretization procedure and an associated step size. A typical case is a point mass moving according to Newton's second law with a time step Δt. Smaller values of Δt generally yield more accurate results (provided that roundoff errors are not a problem) at the cost of increased computational time. In the vibrating-string problem we have discretized two variables, and so we have two step

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sizes, Δt and Δx, to consider. Although you might expect that smaller step sizes are always better, life is not quite this simple.

For the calculation in Fig. 1 we chose the step sizes so that $\Delta x/\Delta t = c$ (that is, $r=1$). It turns out that algorithm (3) is *exact* for this special ratio of the step sizes. The reason can be appreciated from the physics of the problem. The physics, and hence the wave equation, are local in the sense that a particular piece of the string is affected by only adjacent pieces. Consider a disturbance of the string at spatial location i. According to (3), this disturbance will propagate one spatial unit at the next time step to $i \pm 1$, move to $i \pm 2$ at the following time step, and so on. Thus, the algorithm permits a disturbance to propagate at a speed of $\Delta x/\Delta t$. It is clearly good if this speed exactly matches the speed of all solutions. We are thus not surprised that the algorithm works best for $\Delta x/\Delta t = c$, and it is not difficult to show that the corrections to (2) vanish for this ratio of the step sizes.

The behavior for other values of r can be understood from similar arguments. If $r = c\Delta t/\Delta x$ is less than unity, the algorithm will permit a disturbance to travel faster than c, which might be cause for concern. However, it turns out that the amplitude of the disturbance that travels at the maximum speed c/r allowed by the algorithm is very small. It can be shown from (3) that this amplitude is smaller than the amplitude of the "true" disturbance (the one that moves with a speed c) by an amount that decreases by a factor of r^2 after each time step, and thus this amplitude rapidly becomes negligible. Hence, the algorithm will still perform acceptably (although not be exact) when $r<1$. In the regime $r>1$, a similar argument implies that the fastest component *grows* by a factor of r^2 after each time step, and thus the numerical solution rapidly becomes unstable. The algorithmic requirement that the parameter r be less than or equal to unity is known as the *Courant condition*.[2] Its physical origin is particularly easy to appreciate in the context of the wave equation, and our argument also suggests how to adapt this algorithm to treat variants of (1), such as the case of a stiff string.[1]

Tone of a plucked guitar string

To understand the nature of the tone produced by our simulated guitar, we must consider how the vibration of the string is converted into sound. Because the radius of a typical string is much smaller than the wavelength of any sound it might produce, the amount of sound generated by the direct interaction of a string with the surrounding air is negligibly small.[3] For an acoustic guitar, sound is produced by the vibration of the body of the instrument. One end of the string is terminated at the bridge, which is attached to the top plate of the guitar (see Fig. 2). The force from the string on the bridge causes motion of the entire guitar body, which in turn produces sound. To treat this problem in full would require that we model the vibrations of the body, a problem too ambitious to tackle here. Instead, we make the rather drastic assumption that both the body motion and the sound pressure amplitude are proportional to the force on

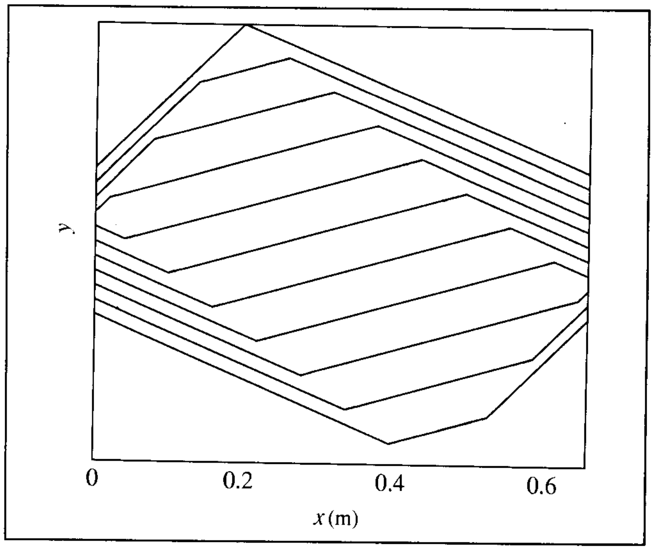

Figure 1. Waves propagating on a string with fixed ends. The string has a length of 0.65 m with c=200 m/s, as would be appropriate for a guitar string. The simulation used Δx=0.001 m, and $\Delta t = \Delta x/c$. The initial string profile is at the top. Successive traces from top to bottom show the string at progressively later times. For clarity, each trace is shifted downward from the previous one.

the bridge in the direction perpendicular to the plane of the top plate of the guitar (the plate to which the bridge is attached). The idea is that the top plate acts like a large speaker, and so it is the perpendicular motion that produces sound most efficiently. This force is equal to the component of the force from the string in the perpendicular direction, which is the tension in the string times $\partial y/\partial x$ (the slope of the string) at the bridge. This quantity can be readily calculated from the simulation, and its behavior for the plucked guitar string is shown in Fig. 2.

The power spectrum of the bridge force is also shown in Fig. 2. As expected, we see a harmonic pattern of peaks at frequencies nf_1, where $f_1 \approx 150$ Hz is the fundamental and n is an integer. The force on the bridge is essentially a square wave (which should be no surprise given the waveforms shown in Fig. 1), and so the force has substantial strength at high frequencies, which causes the amplitudes of the harmonics to fall rather slowly with n. Note that these amplitudes do not vary monotonically with n, and some are very weak (such as the $n=10$ harmonic, which would be near 1500 Hz). This behavior can be understood in terms of the symmetry of the initial string profile as discussed in Ref. 1.

Nature of a piano tone

Our calculation for a plucked guitar string did not really require a numerical solution; the results could have been obtained from the original plucked wave form with a little Fourier analysis. However, the situation for a piano makes a simulation imperative. A piano string is set into motion by the blow from a wooden mallet covered with a compressible layer of felt. Although our first urge might be to treat the felt as a simple spring describable by Hooke's law, it turns out that the physics is not this simple. Experiments have shown that the restoring force for felt depends on the rate at which it is compressed and the preceding compression history. Because of this hysteresis, a full treatment of the piano-hammer problem has not been worked out. Here we shall describe an approximate approach, which is essentially equivalent to the best that has been done to date.[4]

If we ignore the hysteresis, the force of a piano hammer on a string is given approximately by $F_{\text{hammer}}=Kz^p$, where z is the amount that the felt is compressed. The exponent p has a very un-Hooke-like value of $p\sim 2.5$, although p depends somewhat on the way the felt is processed. We can add the hammer to the simulation by treating it as a mass that strikes the simulated string at a certain location, with the hammer-string interaction force F_{hammer}. To implement this scheme, we rewrite the wave equation (1) in a form that emphasizes its connection with Newton's second law:

$$\Delta m\,\frac{\partial^2 y}{\partial t^2}=c^2\Delta m\,\frac{\partial^2 y}{\partial x^2}+F_h(x). \tag{4}$$

The left-hand side of (4) is the mass times the acceleration for an element of the string of length Δx and mass $\Delta m=\rho_{\text{string}}\Delta x\pi\sigma^2$, where σ is the radius of the string. The first term on the right is the restoring force due to neighboring elements of the string, and the second term is the force due to the hammer given by

$$F_h(x)=g(x)F_{\text{hammer}}=g(x)Kz^p, \tag{5}$$

where $g(x)$ is a function that describes how the hammer force is distributed along the string.

The quantities $g(x)$ and z need explanation. We shall take the hammer strike point to be at a location $L_{\text{string}}/7$ from one end of the string (a value similar to that used in most pianos). Because a hammer is not a point object, the force will be distributed to neighboring locations as well. This distribution can be modeled by letting $g(x)$ be a Gaussian function centered at the $x=L_{\text{string}}/7$, with a full width of 1 cm (a typical value). The variable z is the amount that the felt is compressed; it depends on the hammer displacement, y_{hammer}, and the string displacement at the hammer position, $z=y_{\text{hammer}}-y_{\text{string}}$. The hammer displacement is governed by Newton's second law:

$$m_{\text{hammer}}\frac{d^2 y_{\text{hammer}}}{dt^2}=F_{\text{hammer}} \tag{6}$$

with the hammer force given above.

If we write the derivatives in (4) as finite differences, we can derive a difference equation similar to (3). This relation can be used to calculate the motion of the string, and a simple Euler method can be employed for the hammer (6). The strategy is to begin at $t=0$ with the string at $y=0$ everywhere and with an initial velocity for the hammer. When the hammer meets the string, the hammer felt is compressed by an amount equal to the difference between the hammer position and the string position at the strike point. This compression results in an interaction force that acts on both the string and the hammer, causing the string to move and the hammer to rebound. Some results from such a simulation are given in Fig. 3, which shows the string profile during the initial hammer-string impact (0.3 ms), just after the hammer falls away from the string (2.1 ms), and while the string is vibrating freely (3.0 ms). We have used string and hammer parameters appropriate for the note middle C on a piano (see the caption for Fig. 3 for the parameter values). The calculated hammer-string contact time is 2.1 ms, which agrees well with measured values.

As with a guitar, this string vibration leads to sound through the motion it causes at the bridge. In a piano, the strings are attached to a bridge, which is in turn supported by the soundboard. Roughly speaking, the soundboard plays the role of the top plate of the guitar, as sketched in

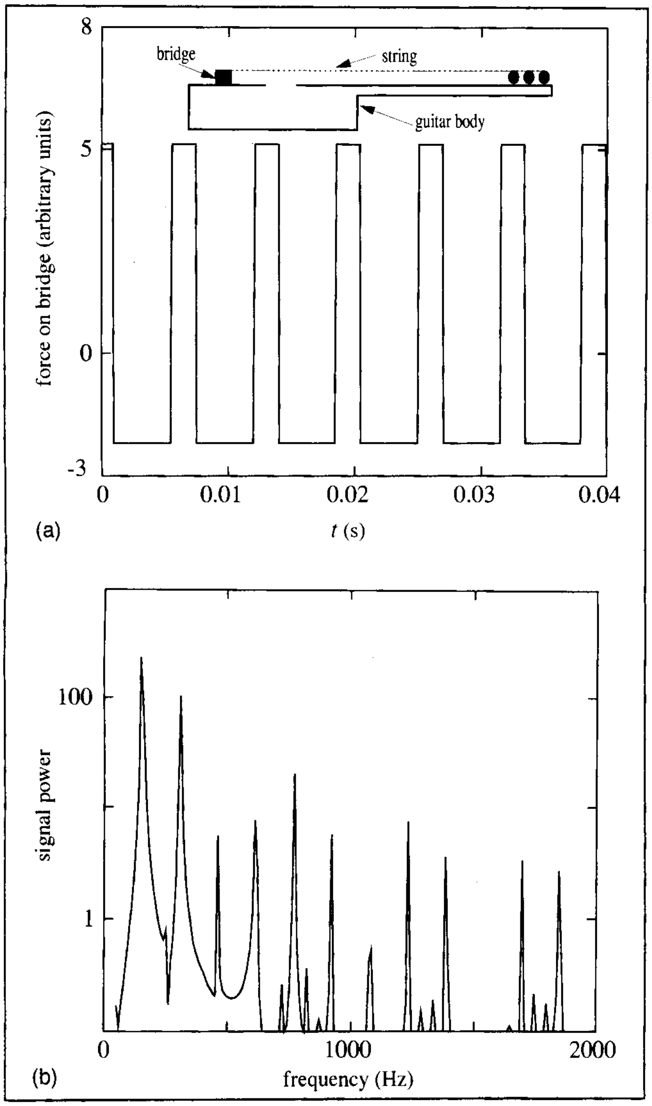

Figure 2. (a) The force on the bridge of our numerical guitar as a function of time from simulation shown in Fig. 1. The inset shows a schematic of an acoustic guitar. (b) The power spectrum of the bridge force.

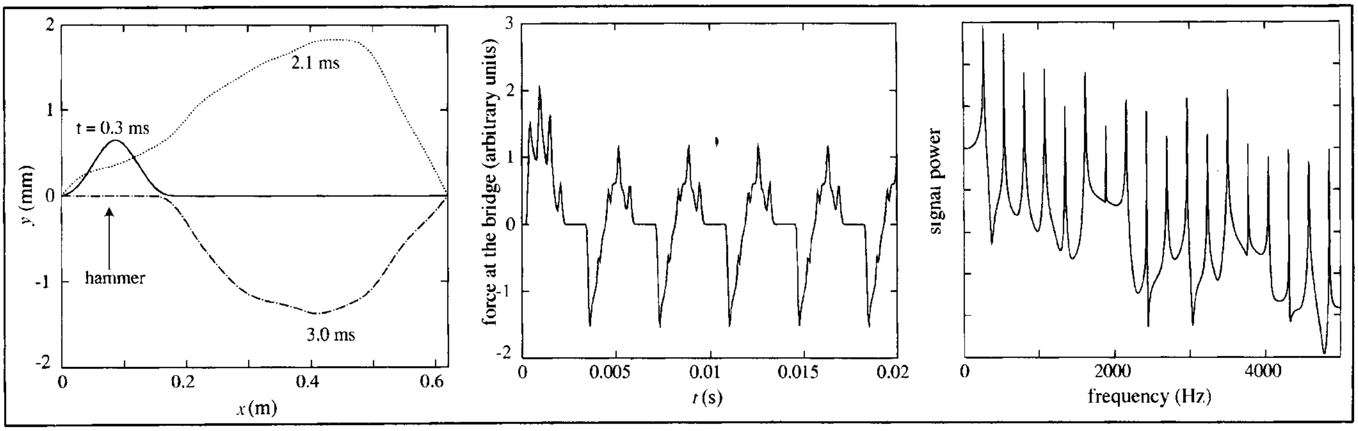

Figure 3. Simulation of a piano string. Left: string profiles at three different times. Middle: the force on the bridge as a function of time. Right: the spectrum of the bridge force. The vertical scale is logarithmic and arbitrary; each tick mark is an order of magnitude. The string and hammer parameters (after Ref. 4) are appropriate for a string near middle C played mezzo forte*:* $L_{\text{string}}=0.62$ *m,* $T=670$ *N,* $\mu=0.006$ *kg/m,* $m_{\text{hammer}}=0.003$ *kg,* $p=2.5$*,* $K=5\times10^9$ *N/m*$^{1/p}$*, and* $\Delta x=L_{\text{string}}/100$*. The initial hammer velocity was 2 m/s, and the hammer struck the string at a distance* $6L_{\text{string}}/7$ *from the bridge.*

Fig. 2. The force on the bridge of our simulated piano has a more complex form than we found for a guitar, demonstrating the crucial importance of the method used to excite the string. This form is also reflected in the spectrum of the piano-string signal, as the high harmonics are somewhat weaker than found in the guitar spectrum. This difference is one of several reasons why the two instruments sound different.

Damping and nonlinearity

The string vibrations we have encountered so far lack an important feature: they do not decay with time. Our simulations have not included any energy-loss mechanisms, an assumption that is not very realistic. For guitar and piano strings we know that the sound typically decays within a few seconds, and the nature of this decay is important for our perception of these tones. There are several effects that contribute to the damping of string vibrations in these instruments. We now consider two sources of damping that illustrate different physical mechanisms.

One source of damping is associated with air drag on the string. Stokes considered this problem nearly 150 years ago and calculated the drag force on a cylinder moving through a fluid.[5] He was interested in the drag force due to the viscosity of the fluid and found that this force is proportional to the velocity. However, for a typical piano string a larger contribution to the drag arises from a different mechanism that yields a drag force that varies more nearly as the square of the velocity. This order-v^2 contribution to the drag force is associated with the kinetic energy imparted to the air as it is pushed by the string and is a well known result[1] for objects moving at speeds of typically 1–30 m/s. Interestingly, while the effect of the viscous force calculated by Stokes has been discussed for stringed instruments,[3] it appears that no one, until now, has estimated the (larger) effect of the drag force, which is quadratic in v.

The corresponding simulation is straightforward; we only need to add a damping term to the wave equation. It is convenient to include the damping force using Newton's second law as we did in (4). We find

$$\Delta m\,\frac{\partial^2 y}{\partial t^2}=c^2\Delta m\,\frac{\partial^2 y}{\partial x^2}+F_h(x)+F_{\text{drag}}, \tag{7}$$

where the last term on the right is the force due to air drag on an element of the string. It has the form[1] $F_{\text{drag}} \approx -C\rho_{\text{air}}Av^2$, where ρ_{air} is the density of air, $A=2\sigma\Delta x$ is the area of an element of the string, which determines how much air is displaced by the string, and C is a constant, which has a value near 1/2. We have repeated the simulation of Fig. 3 with this drag force included. The resulting slow decay of the string vibration can be seen from the time dependence of the root-mean-square vibrational amplitude, which is shown in Fig. 4. We see that the calculated tone persists for many seconds, which is much longer than found with a real piano. Not surprisingly, we shall see that another damping mechanism is more important (that is, leads to a faster decay). Even so, an interesting feature of the damping due to air drag is that it is nonlinear, which makes a numerical approach necessary. This nonlinearity causes different frequencies to be damped at different rates, and so the spectrum of the tone changes with time. It also causes the decay to be nonexponential.

Although we have already touched briefly on the force of the string on the bridge and its role in sound production, we have so far assumed that the bridge is perfectly rigid in our simulations. Of course, this assumption cannot be strictly true, because the bridge must move at least a little if any sound is to be generated. Hence, our simulation should allow the bridge end of the string to move. The work done by the string on the bridge causes a transfer of energy to the bridge and is a mechanism for energy loss. A complete simulation would treat the vibrations of the bridge and the board on which it is mounted. Such a treatment would be very ambitious and has been carried out for a guitar,[6] but not (yet) for a piano. Here we will treat the bridge motion in a simple, but reasonably accurate, way.

To proceed, we need an equation of motion for the bridge. Our first thought might be to treat it as a simple mass, or a mass connected to a spring, but a better approach involves a quantity known as the mechanical impedance, Z. As with the electrical impedance, Z is useful in discussions of the mechanical response in the frequency domain. If the force on the bridge has a single frequency, we can write

$$v_{\text{bridge}} = \frac{F_{\text{bridge}}}{Z}. \tag{8}$$

In general, Z is complex, which is a way of saying that the force on the bridge and its velocity are not in phase with one another. In most cases Z is also a function of frequency, and so (8) is not very useful as an equation of motion in the time domain. However, it turns out that the behavior of Z for a real piano can be used to our advantage. Experiments[7] show that at frequencies from about 100 Hz up to several kilohertz, Z for a piano is approximately independent of frequency and has a value near 10^3 kg/s. If we make the assumption that Z is real, v_{bridge} and F_{bridge} will always be in phase, and the two will be proportional to each other, even in the time domain. With these assumptions we can use (8) as an equation of motion for v_{bridge}, and thus obtain the bridge displacement as a function of time. The bridge displacement is also the displacement of one end of the string. We note again that because the force on the bridge and the bridge velocity are in phase, energy will be transferred from the string to the bridge. Adding the bridge motion to our model gives the result shown in Fig. 4. The string decay is seen to be exponential, with a decay time of about 1 s. This result is in good accord with the decay time found for real piano tones.[3]

For a piano and an acoustic guitar, damping due to bridge motion is more important than the effect due to air drag.

For a piano and an acoustic guitar, damping due to bridge motion is more important than the effect due to air drag, but this mechanism may not be important for all instruments. For example, the end supports in an electric guitar are much more rigid than those of an acoustic instrument, and so air damping may be important in that case. We also mention that many other effects contribute to the decay of piano and guitar tones. In fact, this decay is often *not* purely exponential, and so our model is not complete, although it is a reasonable start.

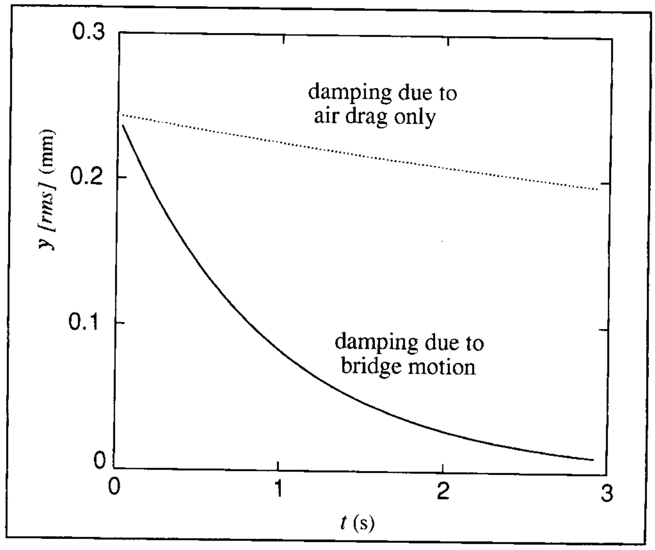

Figure 4. Damping of a piano string: the root-mean-square vibration amplitude as a function of time. The parameter values are σ=0.5 mm, Z=1000 kg/s, ρ_{air}=1.2 kg/m^3, and a drag coefficient C=1/2. The other string and hammer parameters are the same as in Fig. 3.

Longitudinal string vibrations

In our discussion of damping we found that air drag gives rise to a nonlinear term in the wave equation. However, this nonlinear term was found to have a fairly small effect on the motion of a piano string. It turns out that there is another source of nonlinearity, and this makes a very perceptible contribution to the sound produced by guitars and pianos. Our original wave equation (1) was derived assuming that the perpendicular string displacement y is small. This assumption allows us to treat the tension T as a constant. Although this assumption is true to lowest order in y, the tension must increase when the string is displaced from its resting state, and furthermore, T will in general vary with position along the string. Because the wave speed is a function of T, this change of T gives rise to a number of nonlinear additions to (1). Moreover, a transverse displacement of the string can produce *longitudinal* string motion, which has the form of a compressional wave (much like a sound wave). Before we proceed to include this nonlinearity in our simulation, we point out that its effects can be readily heard for a guitar. The initial "twang" of a guitar tone, which is especially noticeable when a string is plucked forcefully, is due to this increase in the tension.

Deriving the nonlinear wave equation for a string requires some care so that the higher-order (nonlinear) terms are included consistently. It turns out that to second order, that is, when terms that are both linear and quadratic in the transverse string displacement and its derivatives have been included, the wave equation for the transverse displacement is identical to our original wave equation (1). However, for the longitudinal displacement we find the wave equation[8]

$$\frac{\partial^2 w}{\partial t^2} = c_\ell^2 \frac{\partial^2 w}{\partial x^2} + \frac{c_\ell^2}{2}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)^2, \tag{9}$$

where w is the longitudinal displacement of the string from its equilibrium position and c_ℓ is the longitudinal wave speed. (Note that here we also assume that $c_\ell \gg c$, which is well satisfied, by an order of magnitude or more, for a typical string.) Equation (9) is similar in form to (1) except for the last term on the right, which is a nonlinear function of the transverse displacement y. The origin of this term can be understood from Fig. 5. The tension at a point along the string is given by

$$T = T_0 + \frac{EA_s ds}{dx}, \tag{10}$$

where T_0 is the nominal tension, E is Young's modulus, $A_s = \pi\sigma^2$ is the cross-sectional area of the string, and ds is the amount the string is stretched at that point. [Equation (10) follows from the definition of the Young's modulus.] From Fig. 5 we can see that ds is given by

$$ds = \sqrt{dx^2 + dy^2} - dx \approx \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx. \tag{11}$$

The longitudinal force on an element of the string is

$$T(x+dx) - T(x) \approx \frac{\partial T}{\partial x} dx = \frac{EA_s}{2} \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)^2 dx, \tag{12}$$

where we have kept terms only to second order in y and its derivatives. If we use the fact that $c_\ell = \sqrt{EA_s/\mu}$, we obtain the nonlinear term in (9).

The longitudinal string motion can be added to our simulation by treating $w(i,n)$ in much the same way as we treated $y(i,n)$. We can write (9) in a finite-difference form and solve for $w(i,n+1)$ in terms of the longitudinal and transverse displacements at earlier times. The result is similar to (3), with the (nonlinear) terms involving $y(i,n)$ acting as an effective force that drives the longitudinal motion. One slight complication concerns the question of spatial and temporal step sizes. We have seen that to ensure numerical stability (and have an exact algorithm) for the transverse wave equation we arrived at the condition $\Delta x/\Delta t = c$,

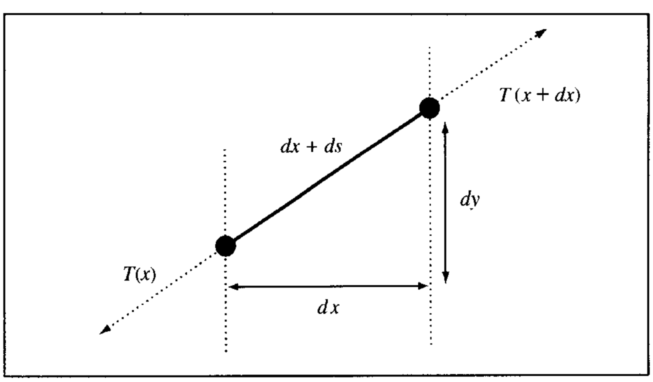

Figure 5. Forces on an element of the string. The increase in the tension is proportional to ds, the amount the string is stretched.

and we are led to a similar condition for the longitudinal wave equation. Because c_ℓ is much greater than c, we are forced to use a much smaller time step to ensure the stability of the longitudinal solution.

There are several ways to deal with this complication; one is to simply use a much smaller time step for the calculation of both the longitudinal and transverse displacements. However, we would then have to give up the good accuracy of our algorithm for the transverse displacement, which is obtained only when the condition $\Delta x/\Delta t = c$ is satisfied. A second approach is to use different time steps for the transverse and longitudinal parts of the problem. That is, we employ a time step $\Delta t = \Delta x/c$ for the calculation of y and a much smaller time step $\Delta t_\ell = \Delta x/c_\ell$ for the calculation of w. Because y enters the longitudinal calculation, we have to be careful how we do this. It is convenient to choose the ratio $\Delta t/\Delta t_\ell$ to be an integer (which is never a poor approximation because $c_\ell/c \gg 1$), so that for each update of y [each iteration of (3)], we update the longitudinal displacement $\Delta t/\Delta t_\ell$ times. Alternatively, we can interpolate (in time) the values of y for use in the equation of motion for w. Both approaches are numerically quite stable.

Some results of such a simulation are shown in Fig. 6. The transverse force on the bridge is the same as shown earlier; it does not change because, as noted above, to second order in the nonlinearities, the transverse wave equation is still (1). It is worth noting that, while the hammer made first contact with the string at $t=0$ in Fig. 6, the transverse pulse did not reach the bridge until $t \sim 1.5$ ms. This is the time required for a transverse disturbance, which moves at the speed c, to travel from the hammer contact point to the end of the string. The new result is for the longitudinal force on the bridge, which exhibits two important features. The first is that this "signal" arrives at the bridge well in advance of the transverse force. Of course, we could have expected this, because the speed of a longitudinal disturbance, c_ℓ, is much greater than c. Hence, the initial sound will be dominated by the longitudinal vibrations. A second important feature of Fig. 6 is that the longitudinal force signal contains a substantial portion of its strength at high frequencies. This also can be understood from the fact that $c_\ell \gg c$. We can be quantitative by comparing the power spectra of the initial 10 ms of the two force signals. The frequency resolution for these spectra (Fig. 6) is not as good as in the spectra of Fig. 3 because we have analyzed only a short segment of the time signal; for this reason the well-defined harmonics discussed earlier are not evident here. The spectrum of the transverse force falls rapidly with frequency, while the longitudinal spectrum is approximately flat out to several kilohertz. Hence, the longitudinal "attack" part of the tone has a different spectral composition than that of the transverse portion.

Our simulation of the longitudinal string vibrations has omitted several important effects. One effect concerns the other transverse mode of the string. It turns out that it too can be excited, and it appears to play an important role in guitars and pianos. In addition, we have ignored the question of how a longitudinal force on the bridge gives rise to sound. For a transverse force the answer is (at a qualitative level) obvious; it causes the bridge and soundboard of the instrument to move like a large speaker. It is not so obvious how a longitudinal force produces sound from such a speaker. In fact, a treatment of this part of the piano problem has not been given (but we are working on it). Nevertheless, the importance of longitudinal string vibrations and their spectral composition are firmly established experimentally,[9] and the essential features of those experimental results are reasonably well described by the simulation we have given here.

This column has touched on only a few of the interesting problems that arise with vibrating strings in musical instruments. Some of these problems involve what might seem, at first glance, to be small modifications of the basic wave equation (1). However, they have a substantial impact on the sound that is produced. After all, most people have no trouble distinguishing a guitar from a piano, and even two pianos do not necessarily sound alike. Hence, a careful treatment of these small modifications is essential. Because of the nonlinear aspects of these problems, and also the complex geometries of most musical instruments, an approach based on computer simulations is extremely useful for dealing with the calculations that arise.

The importance of longitudinal string vibrations and their spectral composition are firmly established experimentally.

Suggestions for further study

(1) Repeat the calculation shown in Fig. 1, but let the ends of the string be completely free. This condition would not be relevant for a vibrating string, but might be applicable to a wind instrument. You should now observe that the reflected waves are not inverted.

(2) In the calculation of Fig. 1 we assumed that the string was held fixed prior to $t=0$ and released from rest. This condition produced left- and right-going kinks, that is, excitations. Set up initial conditions that produce an excitation that travels in only one direction. Hint: You will need to specify the proper initial velocity.

(3) In all our simulations we have assumed that the string is perfectly flexible. Real strings have some stiffness, although it is small for the strings used in musical instruments. The presence of stiffness leads to a modified wave equation of the form

$$\frac{\partial^2 y}{\partial t^2}=c^2\left[\frac{\partial^2 y}{\partial x^2}-\epsilon\frac{\partial^4 y}{\partial x^4}\right], \tag{13}$$

where the stiffness parameter ϵ depends on the Young's modulus, the radius of the string, and other factors. Although ϵ is usually small (a typical value is $\epsilon \sim 1.5\times 10^{-5}$), stiffness has an important effect because it makes the system dispersive. That is, the wave speed now increases slightly as the frequency is increased, and so the string no longer produces a purely harmonic spectrum. This dispersion also upsets the stability of our algorithm, making it necessary to use values of the parameter r less than unity. Calculate the spectrum of a vibrating string described by (13) and determine how much the peaks in the spectrum are shifted from a perfect harmonic sequence. Further details of this calculation (and, in particular, how to deal with the fourth derivative in (13)) are given in Refs. 1, 3, and 4.

(4) To do a better job of modeling an acoustic guitar, we need to treat the bridge more realistically. A full treatment of this motion would involve the vibrations of the body of the guitar, and given its complicated geometry, this problem is very difficult.[6] However, to a rough approximation (which is reasonable at low frequencies), we can model the bridge as an oscillator with several resonant frequencies. For simplicity, assume that it is a single oscillator described by the equation of motion

$$m_b\frac{d^2 y_b}{dt^2}=F-k_b y_b-R_b\frac{dy_b}{dt}, \tag{14}$$

where y_b is the displacement of the bridge, m_b the (effective) mass of the bridge and top plate, F is the force from the string, k_b is the spring constant of the oscillator, and R_b is a damping coefficient. Include this mechanism in the simulation [in place of (8)], and observe how it affects the spectrum. Typical parameters for a guitar are $m_b=0.15$ kg, $k_b=1.5\times 10^5$ N/m, and $R_b=8$ kg/s. The other string parameters were given earlier in connection with our guitar simulation.

(5) A notable feature of the hammer force (5) is that it is a nonlinear function of the amount the felt is compressed, z. Physically the nonlinearity means that the felt becomes effectively stiffer, and the force larger, the more the felt is compressed. The nonlinearity also has an important effect on the piano tone. Explore this effect by comparing the spectrum obtained with a small hammer velocity to that found with a large hammer velocity (values of 0.5 and 4 m/s are realistic choices). You should find that the louder note has more strength in the high harmonics in comparison to the softer note. Hence, a loud note does not just have a larger volume, but its spectrum (that is, its timbre) is also different. Explain

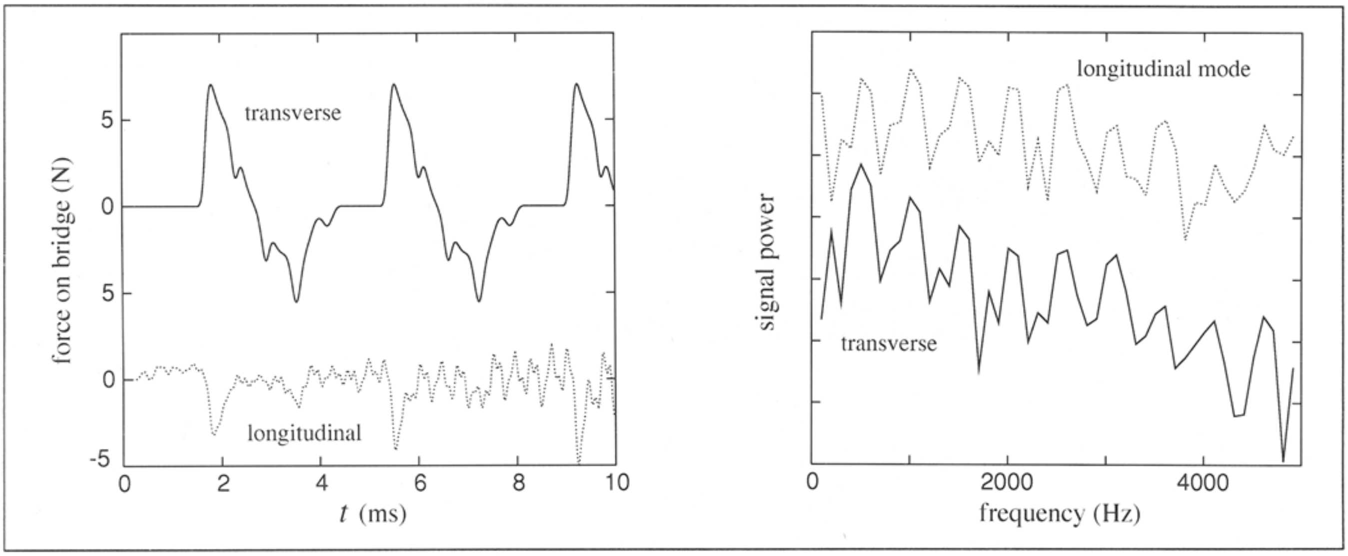

Figure 6. Left: transverse and longitudinal forces for the piano bridge as functions of time. Right: the power spectra of the forces on the left. The vertical scale is logarithmic and arbitrary; each tick mark is an order of magnitude. The Young's modulus was chosen to be 2×10^{11} N/m^2, as is appropriate for a steel string.

this effect in terms of the nonlinearity of the hammer force. It is interesting to listen to the calculated tones. Assume that the sound pressure is proportional to the bridge velocity. To listen to your result, you need to convert the data for the sound pressure as a function of time to a format that can be played on your computer's hardware. On my Sun workstation this conversion can be done with the soundtool program. Another approach is to use a Unix program such as sox (short for "sound exchange") to convert the data to a common sound format such as AIFF, which can be played by one of the many sound utilities available for personal computers. It is convenient to use a sampling rate (Δt^{-1}) of 22 kHz, so that the result matches the rate available with standard hardware.

(6) So far we have considered the motion of only a single string. However, guitars and pianos contain many strings, and in the case of pianos most notes involve three strings. Because all the strings are in contact with the bridge, the bridge motion mediates the interaction between strings. Add a second string to the calculation associated with (8) and Fig. 4. Use a hammer to excite one string and calculate how much motion is induced in the other string. It is also interesting to examine how the vibrational amplitudes of both strings decay with time. Relate your results to the behavior of coupled oscillators.[10]

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From the editors. Please consider submitting a manuscript to the Computer Simulations column. We also invite your comments and suggestions for future columns. Of particular interest are columns such as the present one that show students how to apply their knowledge of basic physics to model realistic problems of current interest. We also are interested in columns that illustrate how to use simple interactive graphics embedded in Fortran 90, C/C++, and Java simulation programs. For further information on submitting a manuscript, visit http://physics.clarku.edu/cip/cip.html.

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